PHY321: Harmonic Oscillations

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Aims and Overarching Motivation

Monday.   Summary on forces and conservation laws, with additional examples. Begin harmonic oscillations.

  Reading suggestion: Taylor chapters 3 and 4 for the summary on forces and conservation laws. Taylor section 5.1 for start harmonic oscillations.

Wednesday.   Harmonic oscillations, basic equations and formalism **Reading suggestions**: Taylor sections 5.1-5.2

Friday.   Harmonic oscillations, damped motion. **Reading suggestion**: Taylor sections 5.3-5.4.

Harmonic Oscillator

The harmonic oscillator is omnipresent in physics. Although you may think of this as being related to springs, it, or an equivalent mathematical representation, appears in just about any problem where a mode is sitting near its potential energy minimum. At that point, $\partial_x V(x) = 0$, and the first non-zero term (aside from a constant) in the potential energy is that of a harmonic oscillator. In a solid, sound modes (phonons) are built on a picture of coupled harmonic oscillators, and in relativistic field theory the fundamental interactions are also built on coupled oscillators positioned infinitesimally close to one another in space. The phenomena of a resonance of an oscillator driven at a fixed frequency plays out repeatedly in atomic, nuclear and high-energy physics, when quantum mechanically the evolution of a state oscillates according to e^{-iEt} and exciting discrete quantum states has very similar mathematics as exciting discrete states of an oscillator.

Harmonic Oscillator, deriving the Equations

The potential energy for a single particle as a function of its position x can be written as a Taylor expansion about some point b (we are considering a one-dimensional problem here)

$$V(x) = V(b) + (x-b)\frac{dV(x)}{dx}|_b + \frac{1}{2!}(x-b)^2\frac{d^2V(x)}{dx^2}|_b + \frac{1}{3!}(x-b)^3 V(x)^{(3)}|_b + \cdots \quad (1)$$

If the position b is at the minimum of the resonance, the first two non-zero terms of the potential are

$$\begin{aligned} V(x) \approx & V(b) + \frac{1}{2!}(x-b)^2\frac{d^2V(x)}{dx^2}|_b, \qquad\qquad (2)\\ = & V(b) + \frac{1}{2}k(x-b)^2, \quad k \equiv \frac{d^2V(x)}{dx^2}|_b, \\ F = & -\frac{dV(x)}{dx} = -k(x-b). \end{aligned}$$

Analyzing the equations

Our equation of motion is, with the only force given by the one-dimensional spring force,

$$m\frac{d^2x}{dt^2} = -kx.$$

Defining the natural frequency $\omega_0^2 = k/m$ we can rewrite this equation as

$$\frac{d^2x}{dt^2} = -\omega_0^2 x.$$

We call this a natural frequency since it is defined by the constants that describe our system, the spring constant k and the mass m of the object.

We can as usual split this equation of motion into one equation for the derivative of the velocity and

$$\frac{dv}{dt} = -\omega_0^2 x,$$

and

$$\frac{dx}{dt} = v.$$

The solution to the equations of motion is given by

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

where A and B are in general complex constants to be determined by the initial conditions.

Checking the Solution

Inserting the solution into the equation of motion we have

$$\frac{d^2 x}{dt^2} = -\omega_0^2 x,$$

we have

$$\frac{d^2 x}{dt^2} = -A\omega_0^2 \cos(\omega_0 t) - B\omega_0^2 \sin(\omega_0 t),$$

and the right-hand side is just $-\omega_0^2 x(t)$. Thus, inserting the solution into the differential equation shows that we obtain the same original differential equation.

Initial condition example

Let us assume that our initial time $t_0 = 0$s and that the initial position $x(t_0) = x_0$ and that $v_0 = 0$ (we skip units here). This gives us

$$x(t = 0) = x_0 = A,$$

and it leaves B undetermined. Taking the derivative of x we obtain the velocity

$$v(t) = -A\omega_0 \sin(\omega_0 t) + B\omega_0 \cos(\omega_0 t),$$

and with

$$v(t = 0) = 0 = B,$$

we see that our solution with these initial conditions becomes

$$x(t) = x_0 \cos(\omega_0 t).$$

Math Digression

From our first homework (exercise 1) we have that (we switch to ω instead of ω_0)

$$\cos(\omega t) = \sum_{n=0}^{\infty} (-1)^n \frac{(\omega t)^{2n}}{(2n)!},$$

and

$$\sin(\omega t) = \sum_{n=0}^{\infty} (-1)^n \frac{(\omega t)^{2n+1}}{(2n+1)!},$$

and that we could write

$$\exp(\pm \imath \omega t) = \cos(\omega t) + \pm \imath \sin(\omega t).$$

This means (show this) that we can write our solution in terms of new constant C and D as

$$x(t) = C \exp(\imath \omega t) + D \exp(-\imath \omega t).$$

To see the relation between these two forms we note that we can write our original solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$ as

$$x(t) = (C + D)\cos(\omega t) + \imath(C - D)\sin(\omega t),$$

meaning that we have $A = C + D$ and $B = \imath(C - D)$.

More Math Manipulations

We can also rewrite the solution in a simpler way. We define a new constant $A = \sqrt{B_1^2 + B_2^2}$ which can be thought as the hypotenuse of a right-angle triangle with sides B_1 and B_2 and $B_1 = A\cos(\delta)$ and $B_2 = A\sin(\delta)$.

We have then

$$x(t) = A\left[\frac{B_1}{A}\cos(\omega t) + \frac{B_2}{A}\sin(\omega t)\right],$$

which becomes

$$x(t) = A\left[\cos(\delta)\cos(\omega t) + \sin(\delta)\sin(\omega t)\right],$$

and using the trigonometric relations for addition of angles we have

$$x(t) = A\cos(\omega t - \delta),$$

where δ is a so-called phase shift.

Energy Conservation

Our energy is given by the kinetic energy and the harmonic oscillator potential energy, that is we have (for a one-dimensional harmonic oscillator potential)

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

We assume that we have initial conditions $v_0 = 0$ (no kinetic energy) and $x(t = 0) = x_0$. With these initial conditions we have

$$x(t) = x_0\cos(\omega_0 t),$$

and the velocity is given by

$$v(t) = -x_0\omega_0\sin(\omega_0 t),$$

The energy is conserved (as we have discussed before) and at $t = t_0 = 0$ we have thus

$$E_0 = \frac{1}{2}kx_0^2.$$

At a time $t \neq 0$ we have

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mx_0^2\omega_0^2\sin^2(\omega_0 t) + \frac{1}{2}kx_0^2\cos^2(\omega_0 t),$$

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Recalling that $\omega_0^2 = k/m$ we get

$$E(t) = \frac{1}{2}kx_0^2 \sin^2(\omega_0 t) + \frac{1}{2}kx_0^2 \cos^2(\omega_0 t) = \frac{1}{2}kx_0^2 = E_0.$$

Energy is thus conserved

The mathematical pendulum

Note: Figure to be inserted.

We consider a pendulum of length l attached to the roof as illustrated in the figure (see handwritten notes from Wednesday Feb 24).

The pendulum consists of a rod and a small object attached to the rod. The mass of this object is m and it is the motion of this object we are concerned with. The distance from the object to the roof is $\boldsymbol{r}$ and we have $|\boldsymbol{r}| = l$.

The angle between the y-axis and the rod is ϕ. The forces at play are the gravitational force and a tension force from the rod to the object. The net for is

$$\boldsymbol{F}^{\text{net}} = \boldsymbol{T} + \boldsymbol{G} = T\sin(\phi)\boldsymbol{e}_1 + T\cos(\phi)\boldsymbol{e}_2 - mg\boldsymbol{e}_2,$$

and with

$$\boldsymbol{r} = l\sin(\phi)\boldsymbol{e}_1 + l\cos(\phi)\boldsymbol{e}_2,$$

the equation of motion becomes

$$m\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{T} + \boldsymbol{G} = T\sin(\phi)\boldsymbol{e}_1 + T\cos(\phi)\boldsymbol{e}_2 - mg\boldsymbol{e}_2.$$

Finding the equations for the x- and y-directions

Using the chain rule we can find the first derivative of $\boldsymbol{r}$

$$\frac{d\boldsymbol{r}}{dt} = l\frac{d\phi}{dt}\cos(\phi)\boldsymbol{e}_1 - l\frac{d\phi}{dt}\sin(\phi)\boldsymbol{e}_2,$$

and thereafter the second derivative in the x-direction as

$$\frac{d^2\boldsymbol{r}}{dt^2}\boldsymbol{e}_1 = l\frac{d^2\phi}{dt^2}\cos(\phi) - l(\frac{d\phi}{dt})^2\sin(\phi),$$

and in the y direction

$$\frac{d^2\boldsymbol{r}}{dt^2}\boldsymbol{e}_2 = -l\frac{d^2\phi}{dt^2}\sin(\phi) - l(\frac{d\phi}{dt})^2\cos(\phi).$$

Collecting terms

We can now set up the equations of motion in the x and y directions and get for the x-direction

$$ml\frac{d^2\phi}{dt^2}\cos(\phi) - ml(\frac{d\phi}{dt})^2\sin(\phi) = T\sin(\phi),$$

and for the y-direction

$$-ml\frac{d^2\phi}{dt^2}\sin{(\phi)} - ml(\frac{d\phi}{dt})^2\cos{(\phi)} = T\cos{(\phi)} - mg.$$

This looks ugly!

Let us rewrite

$$ml\frac{d^2\phi}{dt^2}\cos{(\phi)} = \left[ml(\frac{d\phi}{dt})^2 + T\right]\sin{(\phi)},$$

and

$$-ml\frac{d^2\phi}{dt^2}\sin{(\phi)} + mg = \left[ml(\frac{d\phi}{dt})^2 + T\cos{(\phi)}\right].$$

Still not so nice.

Simple trick

How can we simplify the above equations, rewritten here

$$ml\frac{d^2\phi}{dt^2}\cos{(\phi)} = \left[ml(\frac{d\phi}{dt})^2 + T\right]\sin{(\phi)},$$

and

$$-ml\frac{d^2\phi}{dt^2}\sin{(\phi)} + mg = \left[ml(\frac{d\phi}{dt})^2 + T\right]\cos{(\phi)}.$$

We multiply the first equation with $\cos\phi$ and the second one with $\sin\phi$ and then subtract the two equations. We get then

$$-ml\frac{d^2\phi}{dt^2}(\cos{(\phi)})^2 - ml\frac{d^2\phi}{dt^2}(\sin{(\phi)})^2 + mg\sin{(\phi)} = 0,$$

leading to

$$ml\frac{d^2\phi}{dt^2} = -mg\sin{(\phi)}.$$

We are almost there.

Last step

We divide by m and l and we have the famous non-linear in ϕ (due to the sine function) equation for the pendulumn

$$\frac{d^2\phi}{dt^2} = -\frac{g}{l}\sin{(\phi)}.$$

Introducing the natural frequency $\omega_0^2 = g/l$ we can rewrite the equation as

$$\frac{d^2\phi}{dt^2} = -\omega_0^2\sin{(\phi)}.$$

If we now assume that the angle is very small, we can approximate $\sin(\phi) \approx \phi$ and we have essentially the same equation as we had for harmonic oscillations, that is

$$\frac{d^2\phi}{dt^2} = -\omega_0^2 \phi.$$

The solution to this equation is again given by

$$\phi(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

For the general case, we have to resort to numerical solutions.

Damped Oscillators

We consider only the case where the damping force is proportional to the velocity. This is counter to dragging friction, where the force is proportional in strength to the normal force and independent of velocity, and is also inconsistent with wind resistance, where the magnitude of the drag force is proportional the square of the velocity. Rolling resistance does seem to be mainly proportional to the velocity. However, the main motivation for considering damping forces proportional to the velocity is that the math is more friendly. This is because the differential equation is linear, i.e. each term is of order x, $\dot{x}$, $\ddot{x}\cdots$, or even terms with no mention of x, and there are no terms such as x^2 or $x\ddot{x}$. The equations of motion for a spring with damping force $-b\dot{x}$ are

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{3}$$

Harmonic Oscillator, Damping

Just to make the solution a bit less messy, we rewrite this equation as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \beta \equiv b/2m, \ \omega_0 \equiv \sqrt{k/m}. \tag{4}$$

Both β and ω have dimensions of inverse time. To find solutions (see appendix C in the text) you must make an educated guess at the form of the solution. To do this, first realize that the solution will need an arbitrary normalization A because the equation is linear. Secondly, realize that if the form is

$$x = Ae^{rt} \tag{5}$$

that each derivative simply brings out an extra power of r. This means that the Ae^{rt} factors out and one can simply solve for an equation for r. Plugging this form into Eq. (4),

$$r^2 + 2\beta r + \omega_0^2 = 0. \tag{6}$$

Harmonic Oscillator, Solutions of Damped Motion

Because this is a quadratic equation there will be two solutions,

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{7}$$

We refer to the two solutions as r_1 and r_2 corresponding to the $+$ and $-$ roots. As expected, there should be two arbitrary constants involved in the solution,

$$x = A_1 e^{r_1 t} + A_2 e^{r_2 t}, \tag{8}$$

where the coefficients A_1 and A_2 are determined by initial conditions.

The roots listed above, $\sqrt{\omega_0^2 - \beta_0^2}$, will be imaginary if the damping is small and $\beta < \omega_0$. In that case, r is complex and the factor ert will have some oscillatory behavior. If the roots are real, there will only be exponentially decaying solutions. There are three cases:

Underdamped: $\beta < \omega_0$

$$
\begin{aligned}
x &= A_1 e^{-\beta t} e^{i\omega' t} + A_2 e^{-\beta t} e^{-i\omega' t}, \quad \omega' \equiv \sqrt{\omega_0^2 - \beta^2} \\
&= (A_1 + A_2) e^{-\beta t} \cos \omega' t + i(A_1 - A_2) e^{-\beta t} \sin \omega' t.
\end{aligned} \tag{9}
$$

Here we have made use of the identity $e^{i\omega' t} = \cos \omega' t + i \sin \omega' t$. Because the constants are arbitrary, and because the real and imaginary parts are both solutions individually, we can simply consider the real part of the solution alone:

$$
\begin{aligned}
x &= B_1 e^{-\beta t} \cos \omega' t + B_2 e^{-\beta t} \sin \omega' t, \\
\omega' &\equiv \sqrt{\omega_0^2 - \beta^2}.
\end{aligned} \tag{10}
$$

Critical dampling: $\beta = \omega_0$

In this case the two terms involving r_1 and r_2 are identical because $\omega' = 0$. Because we need to arbitrary constants, there needs to be another solution. This is found by simply guessing, or by taking the limit of $\omega' \to 0$ from the underdamped solution. The solution is then

$$x = A e^{-\beta t} + B t e^{-\beta t}. \tag{11}$$

The critically damped solution is interesting because the solution approaches zero quickly, but does not oscillate. For a problem with zero initial velocity, the solution never crosses zero. This is a good choice for designing shock absorbers or swinging doors.

Overdamped: $\beta > \omega_0$

$$x \quad = \quad A_1 \exp -(\beta + \sqrt{\beta^2 - \omega_0^2})t + A_2 \exp -(\beta - \sqrt{\beta^2 - \omega_0^2})t \qquad (12)$$

This solution will also never pass the origin more than once, and then only if the initial velocity is strong and initially toward zero.

Given b, m and ω_0, find $x(t)$ for a particle whose initial position is $x = 0$ and has initial velocity v_0 (assuming an underdamped solution).

The solution is of the form,

$$
\begin{aligned}
x \quad &= \quad e^{-\beta t} \left[A_1 \cos(\omega' t) + A_2 \sin \omega' t \right], \\
\dot{x} \quad &= \quad -\beta x + \omega' e^{-\beta t} \left[-A_1 \sin \omega' t + A_2 \cos \omega' t \right]. \\
\omega' \quad &\equiv \quad \sqrt{\omega_0^2 - \beta^2}, \quad \beta \equiv b/2m.
\end{aligned}
$$

From the initial conditions, $A_1 = 0$ because $x(0) = 0$ and $\omega' A_2 = v_0$. So

$$x = \frac{v_0}{\omega'} e^{-\beta t} \sin \omega' t.$$

Sinusoidally Driven Oscillators

Here, we consider the force

$$F = -kx - b\dot{x} + F_0 \cos \omega t, \qquad (13)$$

which leads to the differential equation

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = (F_0/m) \cos \omega t. \qquad (14)$$

Harmonic Oscillator, Solutions

Consider a single solution with no arbitrary constants, which we will call a *particular solution*, $x_p(t)$. It should be emphasized that this is **A** particular solution, because there exists an infinite number of such solutions because the general solution should have two arbitrary constants. Now consider solutions to the same equation without the driving term, which include two arbitrary constants. These are called either *homogenous solutions* or *complementary solutions*, and were given in the previous section, e.g. Eq. (10) for the underdamped case. The homogenous solution already incorporates the two arbitrary constants, so any sum of a homogenous solution and a particular solution will represent the *general solution* of the equation. The general solution incorporates the two arbitrary constants A and B to accommodate the two initial conditions. One could have picked a different particular solution, i.e. the original particular solution plus any homogenous solution with the arbitrary constants A_p and B_p chosen at will. When one adds in the homogenous solution, which has adjustable constants

with arbitrary constants A' and B', to the new particular solution, one can get the same general solution by simply adjusting the new constants such that $A' + A_p = A$ and $B' + B_p = B$. Thus, the choice of A_p and B_p are irrelevant, and when choosing the particular solution it is best to make the simplest choice possible.

Harmonic Oscillator, Particular Solution

To find a particular solution, one first guesses at the form,

$$x_p(t) = D\cos(\omega t - \delta), \tag{15}$$

and rewrite the differential equation as

$$D\left\{-\omega^2\cos(\omega t - \delta) - 2\beta\omega\sin(\omega t - \delta) + \omega_0^2\cos(\omega t - \delta)\right\} = \frac{F_0}{m}\cos(\omega t). \tag{16}$$

One can now use angle addition formulas to get

$$D\left\{(-\omega^2\cos\delta + 2\beta\omega\sin\delta + \omega_0^2\cos\delta)\cos(\omega t)\right. \tag{17}$$
$$\left. +(-\omega^2\sin\delta - 2\beta\omega\cos\delta + \omega_0^2\sin\delta)\sin(\omega t)\right\} = \frac{F_0}{m}\cos(\omega t).$$

Both the cos and sin terms need to equate if the expression is to hold at all times. Thus, this becomes two equations

$$D\left\{-\omega^2\cos\delta + 2\beta\omega\sin\delta + \omega_0^2\cos\delta\right\} = \frac{F_0}{m} \tag{18}$$
$$-\omega^2\sin\delta - 2\beta\omega\cos\delta + \omega_0^2\sin\delta = 0.$$

After dividing by $\cos\delta$, the lower expression leads to

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{19}$$

Solving with Driven Oscillations

Using the identities $\tan^2 + 1 = \csc^2$ and $\sin^2 + \cos^2 = 1$, one can also express $\sin\delta$ and $\cos\delta$,

$$\sin\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}, \tag{20}$$
$$\cos\delta = \frac{(\omega_0^2 - \omega^2)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}$$

Inserting the expressions for $\cos\delta$ and $\sin\delta$ into the expression for D,

$$D = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}. \tag{21}$$

For a given initial condition, e.g. initial displacement and velocity, one must add the homogenous solution then solve for the two arbitrary constants. However, because the homogenous solutions decay with time as $e^{-\beta t}$, the particular solution is all that remains at large times, and is therefore the steady state solution. Because the arbitrary constants are all in the homogenous solution, all memory of the initial conditions are lost at large times, $t >> 1/\beta$.

The amplitude of the motion, D, is linearly proportional to the driving force (F_0/m), but also depends on the driving frequency ω. For small β the maximum will occur at $\omega = \omega_0$. This is referred to as a resonance. In the limit $\beta \to 0$ the amplitude at resonance approaches infinity.

Alternative Derivation for Driven Oscillators

Here, we derive the same expressions as in Equations (15) and (21) but express the driving forces as

$$F(t) = F_0 e^{i\omega t}, \tag{22}$$

rather than as $F_0\cos\omega t$. The real part of F is the same as before. For the differential equation,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}, \tag{23}$$

one can treat $x(t)$ as an imaginary function. Because the operations d^2/dt^2 and d/dt are real and thus do not mix the real and imaginary parts of $x(t)$, Eq. (23) is effectively 2 equations. Because $e^{\omega t} = \cos\omega t + i\sin\omega t$, the real part of the solution for $x(t)$ gives the solution for a driving force $F_0\cos\omega t$, and the imaginary part of x corresponds to the case where the driving force is $F_0\sin\omega t$. It is rather easy to solve for the complex x in this case, and by taking the real part of the solution, one finds the answer for the $\cos\omega t$ driving force.

We assume a simple form for the particular solution

$$x_p = De^{i\omega t}, \tag{24}$$

where D is a complex constant.

From Eq. (23) one inserts the form for x_p above to get

$$D\left\{-\omega^2 + 2i\beta\omega + \omega_0^2\right\} e^{i\omega t} = (F_0/m)e^{i\omega t}, \tag{25}$$

$$D = \frac{F_0/m}{(\omega_0^2 - \omega^2) + 2i\beta\omega}.$$

The norm and phase for $D = |D|e^{-i\delta}$ can be read by inspection,

$$|D| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}, \qquad \tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}. \tag{26}$$

This is the same expression for δ as before. One then finds $x_p(t)$,

$$
\begin{aligned}
x_p(t) &= \Re \frac{(F_0/m)e^{i\omega t - i\delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\
&= \frac{(F_0/m)\cos(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.
\end{aligned} \tag{27}
$$

This is the same answer as before. If one wished to solve for the case where $F(t) = F_0 \sin \omega t$, the imaginary part of the solution would work

$$
\begin{aligned}
x_p(t) &= \Im \frac{(F_0/m)e^{i\omega t - i\delta}}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \\
&= \frac{(F_0/m)\sin(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.
\end{aligned} \tag{28}
$$

Damped and Driven Oscillator

Consider the damped and driven harmonic oscillator worked out above. Given F_0, m, β and ω_0, solve for the complete solution $x(t)$ for the case where $F = F_0 \sin \omega t$ with initial conditions $x(t = 0) = 0$ and $v(t = 0) = 0$. Assume the underdamped case.

The general solution including the arbitrary constants includes both the homogenous and particular solutions,

$$x(t) = \frac{F_0}{m} \frac{\sin(\omega t - \delta)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} + A\cos\omega' t\, e^{-\beta t} + B\sin\omega' t\, e^{-\beta t}.$$

The quantities δ and ω' are given earlier in the section, $\omega' = \sqrt{\omega_0^2 - \beta^2}, \delta = \tan^{-1}(2\beta\omega/(\omega_0^2 - \omega^2))$. Here, solving the problem means finding the arbitrary constants A and B. Satisfying the initial conditions for the initial position and velocity:

$$
\begin{aligned}
x(t = 0) = 0 &= -\eta \sin\delta + A, \\
v(t = 0) = 0 &= \omega\eta \cos\delta - \beta A + \omega' B, \\
\eta &\equiv \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}.
\end{aligned}
$$

The problem is now reduced to 2 equations and 2 unknowns, A and B. The solution is

$$A = \eta \sin \delta, \quad B = \frac{-\omega \eta \cos \delta + \beta \eta \sin \delta}{\omega'}. \tag{29}$$

Resonance Widths; the Q factor

From the previous two sections, the particular solution for a driving force, $F = F_0 \cos \omega t$, is

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 \beta^2}} \cos(\omega_t - \delta), \tag{30}$$

$$\delta = \tan^{-1} \left(\frac{2\beta\omega}{\omega_0^2 - \omega^2} \right).$$

If one fixes the driving frequency ω and adjusts the fundamental frequency $\omega_0 = \sqrt{k/m}$, the maximum amplitude occurs when $\omega_0 = \omega$ because that is when the term from the denominator $(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2$ is at a minimum. This is akin to dialing into a radio station. However, if one fixes ω_0 and adjusts the driving frequency one minimize with respect to ω, e.g. set

$$\frac{d}{d\omega} \left[(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2 \right] = 0, \tag{31}$$

and one finds that the maximum amplitude occurs when $\omega = \sqrt{\omega_0^2 - 2\beta^2}$. If β is small relative to ω_0, one can simply state that the maximum amplitude is

$$x_{\max} \approx \frac{F_0}{2m\beta\omega_0}. \tag{32}$$

$$\frac{4\omega^2\beta^2}{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2} = \frac{1}{2}. \tag{33}$$

For small damping this occurs when $\omega = \omega_0 \pm \beta$, so the $FWHM \approx 2\beta$. For the purposes of tuning to a specific frequency, one wants the width to be as small as possible. The ratio of ω_0 to $FWHM$ is known as the *quality* factor, or Q factor,

$$Q \equiv \frac{\omega_0}{2\beta}. \tag{34}$$

Our Sliding Block Code

Here we study first the case without additional friction term and scale our equation in terms of a dimensionless time τ.

Let us remind ourselves about the differential equation we want to solve (the general case with damping due to friction)

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx(t) = 0.$$

We divide by m and introduce $\omega_0^2 = \sqrt{k/m}$ and obtain

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \omega_0^2 x(t) = 0.$$

Harmonic Oscillator, Sliding Block

Thereafter we introduce a dimensionless time $\tau = t\omega_0$ (check that the dimensionality is correct) and rewrite our equation as

$$\frac{d^2x}{d\tau^2} + \frac{b}{m\omega_0}\frac{dx}{d\tau} + x(\tau) = 0,$$

which gives us

$$\frac{d^2x}{d\tau^2} + \frac{b}{m\omega_0}\frac{dx}{d\tau} + x(\tau) = 0.$$

Harmonic Oscillator, Sliding Block, Numerical Aspects

We then define $\gamma = b/(2m\omega_0)$ and rewrite our equations as

$$\frac{d^2x}{d\tau^2} + 2\gamma\frac{dx}{d\tau} + x(\tau) = 0.$$

This is the equation we will code below. The first version employs the Euler-Cromer method.

```python
# Common imports
import numpy as np
import pandas as pd
from math import *
import matplotlib.pyplot as plt
import os

# Where to save the figures and data files
PROJECT_ROOT_DIR = "Results"
FIGURE_ID = "Results/FigureFiles"
DATA_ID = "DataFiles/"

if not os.path.exists(PROJECT_ROOT_DIR):
    os.mkdir(PROJECT_ROOT_DIR)

if not os.path.exists(FIGURE_ID):
    os.makedirs(FIGURE_ID)

if not os.path.exists(DATA_ID):
    os.makedirs(DATA_ID)
```

```python
def image_path(fig_id):
    return os.path.join(FIGURE_ID, fig_id)

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

def save_fig(fig_id):
    plt.savefig(image_path(fig_id) + ".png", format='png')


from pylab import plt, mpl
plt.style.use('seaborn')
mpl.rcParams['font.family'] = 'serif'

DeltaT = 0.001
#set up arrays
tfinal = 20 # in dimensionless time
n = ceil(tfinal/DeltaT)
# set up arrays for t, v, and x
t = np.zeros(n)
v = np.zeros(n)
x = np.zeros(n)
# Initial conditions as simple one-dimensional arrays of time
x0 =  1.0
v0 = 0.0
x[0] = x0
v[0] = v0
gamma = 0.0
# Start integrating using Euler-Cromer's method
for i in range(n-1):
    # Set up the acceleration
    # Here you could have defined your own function for this
    a =  -2*gamma*v[i]-x[i]
    # update velocity, time and position
    v[i+1] = v[i] + DeltaT*a
    x[i+1] = x[i] + DeltaT*v[i+1]
    t[i+1] = t[i] + DeltaT
# Plot position as function of time
fig, ax = plt.subplots()
#ax.set_xlim(0, tfinal)
ax.set_ylabel('x[m]')
ax.set_xlabel('t[s]')
ax.plot(t, x)
fig.tight_layout()
save_fig("BlockEulerCromer")
plt.show()
```

When setting up the value of γ we see that for $\gamma = 0$ we get the simple oscillatory motion with no damping. Choosing $\gamma < 1$ leads to the classical underdamped case with oscillatory motion, but where the motion comes to an end.

Choosing $\gamma = 1$ leads to what normally is called critical damping and $\gamma > 1$ leads to critical overdamping. Try it out and try also to change the initial position and velocity. Setting $\gamma = 1$ yields a situation, as discussed above, where the solution approaches quickly zero and does not oscillate. With zero initial velocity it will never cross zero.